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Invited Article

Bénard convection in liquid crystals

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This article presents a review of investigations concerning the onset of Bénard convective instabilities in layers of nematic and cholesteric liquid crystals. Special emphasis is given to the role of a heat focusing mechanism which gives rise to interesting instability phenomena not seen in conventional isotropic liquids. Theoretical predictions of continuum theory are compared with experimental observations, wherever possible, and suggestions for further investigations are included. Although the linear theory is now well understood, at least for nematics, the non-linear theory has received comparatively little attention. It appears that the development of non-linear analyses for such systems may lead to important information concerning multicritical phenomena and the transition to turbulence in non-equilibrium systems.

1. Introduction

Over the last two decades there has been an increasing interest in liquid crystal research and a considerable advance in our knowledge of a rich variety of novel effects associated with these materials. This is especially true for those of nematic type. Such progress has been largely due to the availability of a viable macroscopic theory and the motivation of technological applications. The books by de Gennes [1] and Chandrasekhar [2] give detailed accounts of the physical properties of these transversely isotropic liquids and the review articles by Ericksen [3] and Leslie [4] provide comprehensive accounts of the continuum theory pertinent to this article.

The aim is to review the investigations concerning the occurrence of Bénard convective instabilities in nematic and cholesteric liquid crystals. The classical Rayleigh–Bénard problem is that in which a sample of newtonian fluid contained between two large horizontal flat plates is subjected to an adverse thermal gradient. Provided the temperature difference between the plates is less than some critical value, the system remains in equilibrium and there is no flow. However, as this critical value is exceeded the onset of stationary convection is observed. This simple hydrodynamic instability occurs when the buoyancy force due to thermal expansion near the lower plate is sufficient to overcome the opposing viscous shear force. The intricate thermo-mechanical coupling between flow, temperature and orientation of the anisotropic axis exhibited in liquid crystals allows the possibility of interesting phenomena not seen in conventional isotropic liquids. In particular, novel instabilities are driven by a heat focusing mechanism which is not available in a classical fluid. As a result threshold values for stationary convection are drastically reduced and instability can occur when heating is from above as well as below. In addition, oscillatory convective instabilities with inverse bifurcation and hysteresis effects are possible.

Several authors have conjectured that thermal gradients can directly influence the orientation of the anisotropic axis or director. For example, in a vertical thermal gradient, Stewart [5] reports that 4,4′-dimethoxyazoxybenzene (PAA) adopts a vertical orientation when heating is from below but a horizontal orientation when heating is from above. Rajan and Picot [6] review such literature on thermal transport phenomena in liquid crystals prior to about 1971. For reasons given by Leslie [4], support for this point of view based on thermal conductivity measurements is somewhat debatable. In fact continuum theory appears to be in conflict with these observations since it yields solutions in which thermal gradients do not influence alignment in nematics. Fortunately investigations of thermally induced convective motion in nematic layers suggest a possible alternative explanation of the experimental observations. Moreover, since predictions of the theory agree rather well with the more recent and better controlled empirical investigations, it seems that unambiguous evidence of thermal gradients influencing the alignment directly is required before a revision of the theory is attempted.

In this article we describe the development of the study of thermal convection in liquid crystals since about 1971, comparing theoretical predictions with the available experimental evidence wherever possible. Since studies of thermal convective instabilities in cholesterics are comparatively few in number, this review, with the exception of §9, is entirely concerned with thermal convection in nematics. Furthermore the number of non-linear analyses is rather small and so, apart from §8, all stability analyses presented here are linear. Finally, unless stated otherwise, numerical results given throughout this article are obtained by employing the available data for the material 4-methoxybenzilidene-4′-*n*-butylaniline (MBBA). This is somewhat regrettable in view of the variety of more stable nematic materials now available but unfortunately virtually all calculations have employed such data.

2. The Ericksen–Leslie equations

Throughout this article it is assumed that the continuum equations governing the behaviour of incompressible nematic and cholesteric liquid crystals are those proposed by Ericksen [7] and Leslie [8, 9, 10]. Their theory derives equations for determining the velocity field **v**, the director **n** and temperature T, **n** being a unit vector denoting the orientation of the anisotropic axis. Employing the notation of Leslie's review article, the appropriate equations in cartesian tensor form are the constraints

$$v_{i,i} = 0, \quad n_i n_i = 1 \tag{2.1}$$

together with the balance laws

$$\varrho \dot{v}_i = -p_{,i} - \left(\frac{\partial W}{\partial n_{k,j}} n_{k,i}\right)_{,j} + \tilde{t}_{ij,j} + F_i, \tag{2.2}$$

$$\sigma \ddot{n}_i = \gamma n_i + \left(\frac{\partial W}{\partial n_{i,j}}\right)_{,j} - \frac{\partial W}{\partial n_i} + \tilde{g}_i + G_i, \tag{2.3}$$

$$T\dot{S} = r - q_{i,i} + \tilde{t}_{ij}A_{ij} - \tilde{g}_i N_i + \alpha e_{ijk}\{n_j(N_k + A_{kp}n_p)\}_{,i}, \quad S = -\partial W/\partial T, \tag{2.4}$$

where a superposed dot denotes a material time derivative, e_{ijk} is the alternating tensor and

$$\tilde{t}_{ij} = \alpha_1 n_k n_p A_{kp} n_i n_j + \alpha_2 n_j N_i + \alpha_3 n_i N_j + \alpha_4 A_{ij} + \alpha_5 n_j n_k A_{ki} + \alpha_6 n_i n_k A_{kj} + \alpha_7 e_{ipq} n_p T_{,q} n_j + \alpha_8 e_{ijq} n_p T_{,q} n_i, \quad (2.5)$$

$$\tilde{g}_i = -\gamma_1 N_i - \gamma_2 A_{ij} n_j - \gamma_3 e_{ijk} n_j T_{,k}, \quad (2.6)$$

$$q_i = \kappa_1 T_{,i} + \kappa_2 n_i n_k T_{,k} + \kappa_3 e_{ijk} n_j N_k + \kappa_4 e_{ijk} n_j A_{kp} n_p, \quad (2.7)$$

$$2A_{ij} = v_{i,j} + v_{j,i}, \quad 2N_i = 2\dot{n}_i + (v_{k,i} - v_{i,k}) n_k. \quad (2.8)$$

Here p and γ are arbitrary functions for the pressure and director tension arising from the constraints (2.1), ϱ is the density and σ is a positive inertial constant. For reasons given by Pieranski, Brochard and Guyon [11], the director inertia term $\sigma\ddot{n}_i$ in equation (2.3) is considered to be negligible. **F** and **G** represent any external body force and generalised body force acting, respectively. In calculations we assume that such forces arise only from gravity or an applied uniform magnetic field so that employing Ericksen's hypotheses [12]

$$\mathbf{F} = -\varrho\mathbf{g}, \quad \mathbf{G} = \chi_a(\mathbf{B}\,.\,\mathbf{n})\mathbf{B}, \quad (2.9)$$

where **g** is the gravitational acceleration, **B** is the magnetic flux density and χ_a is the anisotropic part of the magnetic susceptibility. The stored energy per unit volume W is assumed to take the form

$$2W = 2W_0 + K_1(n_{k,k})^2 + K_2(e_{ijk} n_i n_{k,j} + \tau)^2 + K_3 n_{i,k} n_k n_{i,l} n_l \quad (2.10)$$

and the heat supply function r in equation (2.4) is assumed to be zero throughout this paper. Material parameters arising in the theory depend only upon temperature and must satisfy the conditions

$$\gamma_1 = \alpha_3 - \alpha_2, \quad \gamma_2 = \alpha_6 - \alpha_5, \quad \gamma_3 = a_8 - \alpha_7. \quad (2.11)$$

As is common in calculations, we also employ the constraint proposed by Parodi [13], namely

$$\alpha_2 + \alpha_3 = \alpha_6 - \alpha_5. \quad (2.12)$$

Following Parodi's arguments, Prost [14] has proposed the additional constraints

$$\kappa_3 = \alpha_8 - \alpha_7, \quad \kappa_4 = \alpha_7 + \alpha_8. \quad (2.13)$$

The material parameters α, α_7, α_8, γ_3, κ_3, κ_4 and τ are peculiar to cholesterics and are all identically zero for nematic liquid crystals. Finally we note that the pitch P of a cholesteric is related to τ by

$$P = 2\pi/|\tau|. \quad (2.14)$$

3. The linearized equations for Bénard convection in nematics

We are concerned with the stability of a thin layer of nematic liquid crystal subjected to a vertical thermal gradient. The sample is contained between two large, horizontal flat plates with the upper plate occupying the plane $z = h$ being held at a temperature T_2 while the lower plate occupies the plane $z = 0$ and is held at a temperature T_1. Two particular arrangements with strong anchoring at the boundaries have received most attention. In one a uniformly parallel director orientation obtains at the boundary (the planar problem) while in the other n is perpendicular to the bounding surfaces (the homeotropic problem). Unless stated otherwise it is assumed throughout that any applied magnetic field is aligned parallel to the initial uniform orientation of the director. If $\mathbf{n}$ takes the constant value

$$\mathbf{n}^0 = (\cos\theta_0, 0, \sin\theta_0) \tag{3.1}$$

on both $z = 0$ and $z = h$, a simple equilibrium solution of equations (2.1)–(2.4) is

$$\mathbf{n} = \mathbf{n}^0, \quad \mathbf{v} = \mathbf{0}, \qquad T = T_1 + \beta z, \tag{3.2}$$

where β is the constant temperature gradient $(T_2 - T_1)/\mathrm{h}$. From equation (2.7) we can define a thermal conductivity through the relation

$$q_3 = -k(\theta_0)\beta, \quad k(\theta_0) = -\kappa_1 - \kappa_2 \sin^2\theta_0 \tag{3.3}$$

so that

$$k_\perp = k(0) = -\kappa_1, \quad k_\parallel = k(\pi/2) = -\kappa_1 - \kappa_2, \quad k_a = k_\parallel - k_\perp = -\kappa_2, \tag{3.4}$$

where $k_\parallel$ and $k_\perp$ are the conductivity constants parallel and perpendicular to the director $\mathbf{n}$, and k_a is the anisotropic part of the thermal conductivity. It is this constant k_a that has a significant role in the onset of stationary convection in nematics. Since k_a is positive for all known nematics it is assumed throughout the article, unless stated otherwise, that $k_a > 0$. Physically this means that heat is more readily conducted along rather than perpendicular to the director axis.

Now consider the equilibrium state defined by equations (3.1) and (3.2) to be disturbed by a small amplitude velocity field, $\mathbf{v}$, associated with which is a director field, $\mathbf{n}^0 + \hat{\mathbf{n}}$, and a temperature field, $T + \hat{s}$. In addition there is a pressure field, $p + \hat{p}$, and a director tension field, $-\chi_a B^2 + \hat{\gamma}$, where p and $-\chi_a B^2$ are the respective equilibrium values and B is the magnetic flux density. Adopting the Boussinesq approximation [15] which neglects the variation of material parameters with temperature except where associated with gravity, a linearisation of equations (2.1)–(2.4) yields the system of linear equations

$$v_{i,i} = 0, \quad n_i^0 \hat{n}_i = 0, \tag{3.5}$$

$$\varrho \frac{\partial v_i}{\partial t} = \varrho g \alpha' k_i - \hat{p}_{,i} + A_{ijkm} v_{j,km} + B_{ijk} \frac{\partial}{\partial t} \hat{n}_{j,k}, \tag{3.6}$$

$$\hat{\gamma} n_i^0 - \chi_a B^2 \hat{n}_i + C_{ijkm} \hat{n}_{j,km} + D_{ijk} v_{j,k} - \gamma_1 \frac{\partial \hat{n}_i}{\partial t} = 0, \tag{3.7}$$

$$\frac{\partial \hat{s}}{\partial t} + \beta v_3 = -\kappa_1 \hat{s}_{,ii} - \kappa_2 n_j^0 n_i^0 \hat{s}_{,ij} - \beta\kappa_2 (k_j n_i^0 \hat{n}_{j,i} + k_j n_j^0 \hat{n}_{i,i}), \tag{3.8}$$

for the perturbation variables v, $\hat{n}$, $\hat{s}$, $\hat{p}$ and $\hat{\gamma}$. Here α' is the thermal expansion coefficient, k_i is the unit vector in the z direction and

$$\left.\begin{aligned}
2A_{ijkm} &= 2\alpha_1 n_i^0 n_j^0 n_k^0 n_m^0 + (\alpha_2 + \alpha_5)\delta_{ik} n_j^0 n_m^0 + (\alpha_3 + \alpha_6)\delta_{km} n_i^0 n_j^0 \\
&\quad + \alpha_4 \delta_{ij}\delta_{km} + (\alpha_5 - \alpha_2)\delta_{ij} n_k^0 n_m^0, \\
C_{ijkm} &= K_2 \delta_{ij}\delta_{km} + (K_1 - K_2)\delta_{jk}\delta_{im} + (K_3 - K_2)\delta_{ij} n_k^0 n_m^0, \\
B_{ijk} &= \alpha_2 \delta_{ij} n_k^0 + \alpha_3 \delta_{jk} n_i^0, \quad 2D_{ijk} = (\gamma_1 - \gamma_2)\delta_{ij} n_k^0 - (\gamma_1 + \gamma_2)\delta_{ik} n_j^0,
\end{aligned}\right\} \tag{3.9}$$

where δ_{ij} is the Kronecker delta, and κ_1 and κ_2 now denote thermal diffusivities rather than conductivities. Equations (3.5)–(3.8) are the effective starting points for most of the linear analyses described in this article and they illustrate the rather complicated coupling of perturbation variables that is responsible for driving thermal instabilities not found in isotropic liquids. Strong anchoring of the director at the boundary is usually assumed so that, unless stated otherwise, the boundary conditions on $z = 0$ and $z = h$ are

$$\mathbf{v} = \mathbf{0}, \quad \hat{\mathbf{n}} = \mathbf{0}, \quad \hat{s} = 0. \tag{3.10}$$

4. A one dimensional model

In an initial investigation of the planar problem, Dubois-Violette [16] employs a one dimensional analysis which has the merit of being very simple and yet clearly illustrates the heat focusing mechanism that drives the stationary convective instability in thin layers of nematic liquid crystal. Consider the planar equilibrium configuration to be disturbed by perturbations of the form

$$\hat{\mathbf{n}} = (0, 0, n)\exp(\omega t), \quad \mathbf{v} = (0, 0, v)\exp(\omega t), \quad \hat{s} = s\exp(\omega t) \tag{4.1}$$

where n, v, s are functions of x only. The constraints (3.5) are clearly satisfied identically while the balance equations become

$$\varrho\omega v = \varrho g\alpha' s + (\eta_b/2)\frac{\partial^2 v}{\partial x^2} + \alpha_2\omega\frac{\partial n}{\partial x}, \tag{4.2}$$

$$K_3\frac{\partial^2 n}{\partial x^2} + \left(\frac{\gamma_1 - \gamma_2}{2}\right)\frac{\partial v}{\partial x} - \left(\gamma_1\omega + \chi_a B^2\right)n = 0, \tag{4.3}$$

$$\omega s + \beta v = k_\parallel\frac{\partial^2 s}{\partial x^2} + \beta k_a\frac{\partial n}{\partial x}, \tag{4.4}$$

where $\eta_b = \alpha_4 + \alpha_5 - \alpha_2$.

Setting ω equal to zero and

$$(n, v, s) = (in_0, v_0, s_0)\exp(iqx) \tag{4.5}$$

in equations (4.2)–(4.4), the determinantal condition for a non-trivial solution yields an expression for the critical threshold gradient given by

$$\beta_c = \beta_0/\{1 - (\alpha_2 k_a/K_3)/(1 + \chi_a B^2/K_3 q^2)\}, \tag{4.6}$$

where β_0 is a typical threshold value for an isotropic liquid. For MBBA $\alpha_2 k_a/K_3$ is of order 10^2 and hence, in the absence of an applied magnetic field, threshold values are significantly lower than those for isotropic liquids having similar physical properties. This means that convection may occur in nematic layers whose thickness is considerably less than that necessary for convection in newtonian fluids. The term $\beta k_a \partial n/\partial x$ in the heat conduction equation (4.4) represents a combining of the effects of distortion in alignment with the anisotropy in the thermal diffusivity to produce a so called heat focusing effect. It is the coupling between this term and the viscous torque $\{(\gamma_1 - \gamma_2)/2\}\partial v/\partial x$ in the angular momentum equation (4.3) that drives the instability. Further the sign of k_a would appear to be of crucial importance. Assuming k_a is positive stationary convection is expected to occur in planar samples when heating is from below. However if k_a were negative one would have to heat the upper plate!

An explanation of the destabilizing effects of this heat focusing mechanism is provided by a study of the relaxation times for the various perturbation variables. Examining solutions of the form

$$(v, n, s) = (v_0(t), n_0(t), s_0(t))\exp(iqx), \tag{4.7}$$

the relaxation time constants for fluctuations in flow, orientation and temperature are found to be given by

$$T_v = \varrho/(\eta_b q^2), \quad T_n = \gamma_1/(K_3 q^2 + \chi_a B^2), \quad T_s = (k_{\|} q^2)^{-1}. \tag{4.8}$$

At the threshold, where $\dot{v}_0$, $\dot{n}_0$ and $\dot{s}_0$ all vanish

$$\frac{T_v T_s}{\tau^2}(1 + T_n/T_a) = 1, \quad T_a = -\gamma_1/(k_a \alpha_2 q^2), \quad \tau^2 = (-\alpha' g \beta_c)^{-1}, \tag{4.9}$$

T_a being a characteristic time associated with the destabilizing heat focusing mechanism. We note that $T_v T_s/\tau^2$ is π^4 times the critical Rayleigh number for an isotropic fluid having viscosity η_b and thermal diffusivity $k_{\|}$. For a 1 mm thick layer T_n/T_a is approximately 10^3 and it is this large ratio of characteristic times that is responsible for the substantial decrease in the critical threshold gradient and the corresponding critical Rayleigh number. From equations (4.6) and (4.8) it follows that a magnetic field applied parallel to the initial alignment is stabilizing. However if the field is applied perpendicularly across the sample with flux density B it is destabilizing with B^2 being replaced by $-B_\perp^2$.

Despite the simplicity of the model and the fact that solutions of the form (4.7) do not satisfy the boundary conditions (3.10), these results agree well qualitatively with the observations in planar layers reported by Guyon and Pieranski [17] and by Dubois-Violette, Guyon and Pieranski [18]. At threshold they observe $T_1 - T_2$ to be 2·2°C and 15·5°C in 1 mm and 0·5 mm thick layers, respectively, when B is zero. The associated wavelengths are compatible with the estimate $q = \pi/h$ which when used in the analyses [16, 18] yield results that agree reasonably well quantitatively with experimental observations. The latter paper describes an experimental study of the dependence of β_c upon the field strength, there being good agreement between observations and theoretical calculations using equation (4.6). In particular they find that $|\beta_c|$ increases (decreases) linearly with $B^2(B_\perp^2)$ according to the empirical formulae

$$\Delta T_c(B) = \Delta T(B = 0)\left\{1 + \frac{B^2}{B_c^2}\right\}, \quad \Delta T_c(B_\perp) = \Delta T(B_\perp = 0)\left\{1 - \frac{B_\perp^2}{B_c^2}\right\}, \tag{4.10}$$

where B_c is the Freedericksz transition value $(\pi/h)(K_3/\chi_a)^{1/2}$ and $\Delta T \equiv T_2 - T_1$.

A similar calculation for homeotropic alignment by Rajan [19] yields

$$\beta_c = \beta_0/(1 - \alpha_3 k_a/K_1). \tag{4.11}$$

For MBBA, $|\alpha_3 k_a/K_1|$ is of order one and it follows that $\beta_c \approx \beta_0$ and

$$T_a = (-\gamma_1)/k_a \alpha_3 q^2 \gg (-\gamma_1)/k_a \alpha_2 q^2.$$

This suggests that convection does not occur in homeotropic layers of order 1 mm thickness and observations by Pieranski, Dubois-Violette and Guyon [20] support this conclusion when heating is from below. However, they do observe a convective instability in such thin layers when heating is from above and report results in agreement with an empirical law analogous to that in equation $(4.10)_1$ for an applied stabilizing magnetic field. To describe such phenomena, even qualitatively, it is obvious that a more general analysis than that employed here is required.

5. Stationary convection in planar and homeotropic layers

A variety of methods have been employed in analysing the onset of stationary convection in both planar and homeotropic layers. Typically they examine stability with respect to disturbances of the form

$$\hat{\mathbf{n}} = (0, 0, n,)\exp(imx + \omega t), \quad \mathbf{v} = (u, 0, v)\exp(imx + \omega t),$$
$$\hat{s} = s\exp(imx + \omega t) \tag{5.1}$$

and

$$\hat{\mathbf{n}} = (n, 0, 0)\exp(imx + \omega t), \quad \mathbf{v} = (u, 0, v)\exp(imx + \omega t),$$
$$\hat{s} = s\exp(imx + \omega t) \tag{5.2}$$

respectively, where n, u, v, s are functions of z. To obtain threshold values for the onset of stationary convection, the principle of exchange of stabilities is adopted so that critical values are found by setting $\omega = 0$. For each value of m, a critical value $\beta_c(m)$ is expected and that value having the smallest magnitude is called the critical thermal gradient β_c.

Currie [21] and Dubois-Violette [22] were the first to analyse both problems in some detail. With the aid of several rough approximations concerning the magnitudes of certain material parameters. Currie introduces the change of variable $\zeta = (2z - h)/h$ and reformulates the planar problem as that of solving

$$(D^2 - a^2)^4 v + a^4 M\beta v = 0 \tag{5.3}$$

subject to the conditions

$$v = Dv = (D^2 - a^2)^2 v = (D^2 - a^2)^3 v = 0 \tag{5.4}$$

on $\zeta = \pm 1$, where $M = -h^4 \varrho g\alpha' k_a \gamma_1 / 8\alpha_4 K_2 \kappa_1$, $a = mh/2$ and $D \equiv d/d\zeta$. Employing variational methods and a simple trial function he obtains the instability criterion for PAA as $\Delta Th^3 < -8 \times 10^{-3}\,\mathrm{cm}^3\,{}^\circ\mathrm{C}$. A similar calculation for homeotropic layers yields the instability condition $\Delta Th^3 > 24 \times 10^{-4}\,\mathrm{cm}^3\,{}^\circ\mathrm{C}$. Thus for convection to occur the lower plate must be heated in a planar layer but the upper plate in a homeotropic layer.

Dubois-Violette [22] seeks Fourier mode solutions having a z dependence of the form $\exp(il_j z)$. For a given mode with β and m fixed the condition for the resulting set of linear, homogeneous algebraic equations to have a non-trivial solution results in an eighth order polynomial equation for the variable r_j $(\equiv l_j/m)$. Expressing the perturbation variables as linear combinations of these eight solutions, the modes with wavenumber m which develop in the layer are found by solving the secular equation which results from satisfying the boundary conditions. Since an analytic solution is not practicable, she computes results for MBBA and these are in agreement with the conclusions of Currie [21]. In particular, critical values for a sample thickness of 0·5 mm are found to be

$$\Delta T_p = -21^\circ\mathrm{C}, \quad m_p = 30 \quad \text{and} \quad \Delta T_H = 39{\cdot}8^\circ\mathrm{C}, \quad m_H = 28{\cdot}8, \tag{5.5}$$

where subscripts p and H denote results pertinent to the planar and homeotropic problems, respectively. In the absence of an applied magnetic field, the linear equations indicate that βh^4 is a universal constant so that $\Delta T_p = -2{\cdot}6^\circ\mathrm{C}$ and $\Delta T_H = 5^\circ\mathrm{C}$ when $h = 1$ mm. These theoretical predictions agree well with the experimental observations by Guyon and Pieranski [17] and Dubois-Violette, Guyon and Pieranski [18] in planar samples and Pieranski, Dubois-Violette and Guyon [20] in homeotropic layers.

Barratt and Sloan [23, 24] attempt an analytic solution of the problem by employing a Fourier series method used by Jeffreys [25]. They obtain an expression for the critical gradient in the form of a consistency condition which can be written symbolically as

$$f(a, R) = 0, \tag{5.6}$$

where R is the Rayleigh number. Although this criterion is in the form of an infinite series, the convergence is so fast that an excellent approximation for the threshold is given by retaining only the first term in the series. Their results agree well with those of Dubois-Violette [22] and the available experimental evidence.

Excellent pictorial descriptions of the physical stabilizing and destabilizing forces present in both configurations when heating is from either above or below can be found in [16], [18], [20] and [22]. Here we give only a brief verbal account of the physical process. For both equilibrium configurations, the heat flux vector is perpendicular to the bounding plates. In the event of a sinusoidal fluctuation in alignment, the associated perturbation in heat flux is in the horizontal direction and so the heat flux vector is no longer normal to the plates. This is due entirely to the heat focusing effect and the result is the formation of alternate warmer and cooler regions. Under the influence of buoyancy forces these regions tend to move up or down, respectively. Induced viscous torques Γ_1 and Γ_2, associated with the vertical and horizontal components of the flow field, respectively, then act on the director and it is the resultant viscous torque that either stabilizes or destabilizes the system. In a planar layer

$$\Gamma_1 = \alpha_2 \partial v/\partial x \quad \text{and} \quad \Gamma_2 = \alpha_3 \partial u/\partial z,$$

while in a homeotropic layer

$$\Gamma_1 = -\alpha_2 \partial u/\partial z \quad \text{and} \quad \Gamma_2 = -\alpha_3 \partial v/\partial x.$$

Since $|\alpha_2| \gg |\alpha_3|$ for MBBA it is evident that $|\Gamma_1| \gg |\Gamma_2|$ for circular rolls. The one dimensional model of the previous section therefore includes the dominant viscous torque in the planar problem but neglects it in the homeotropic problem. This explains why the one dimensional model fails to describe the situation in the latter problem while yielding fairly accurate results for the former problem.

Using a Galerkin method [26], Verlarde and Zuniga [27] demonstrate that stationary convection is possible in a planar layer heated from above. This requires $|\Gamma_2| > |\Gamma_1|$ with the appearance of elongated rather than circular rolls. Although ΔT_c is so large as to be impractical for layers of order 1 mm thickness, it might be possible to observe this effect in relatively thick samples. In addition they examine more fully than earlier studies the effect of stabilising magnetic fields upon threshold values. When heating is from below in a planar sample, they find that $|\Delta T_c|$ initially increases linearly with B^2 before becoming asymptotic to a value two or three times smaller than the corresponding value for isotropic liquids. This occurs when $B \approx 0{\cdot}06T$ in a 5 mm layer. For a homeotropic layer heated from above they find that ΔT_c increases rapidly with B until the instability is effectively suppressed by a field of about $0{\cdot}03T$ in a 5 mm layer. Barratt and Manley [28] obtain accurate numerical results for the variation of threshold values with field strength for the planar problem which confirm the predictions of Velarde and Zuniga [27]. However they find that the asymptote value is reached at about $0{\cdot}16T$.

Other attempts at solving these problems include those of Miyakawa [29] and Askar [30]. Miyakawa considers the propagation of infinitesimal disturbances in homeotropic nematics subject to a thermal gradient and predicts the onset of convection from an examination of their behaviour. Askar examines both problems but employs the rather unrealizable free-free boundary conditions in an attempt to simplify calculations.

This analysis restricts consideration to a class of disturbances dependent on only one horizontal spatial variable. Since predictions agree rather well with observations we might question the necessity of examining stability with respect to disturbances that are periodically dependent on both horizontal spatial variables. Fraser [31] and Barratt and Zuniga [32] examine the stability of planar and homeotropic layers with respect to these more general disturbances and obtain results that suggest that the critical threshold values remain unchanged.

Finally, Barratt, Coles and Hodson [33] relax the assumption of an initial uniform thermal gradient and consider the initial temperature to be a linear piecewise function so that

$$dT/dz = \begin{cases} \beta/\delta, & 0 \leqslant z \leqslant \delta h, \\ 0 & \delta h < z \leqslant h, \end{cases} \tag{5.7}$$

where $0 < \delta \leqslant 1$. This may be viewed as a simple model to describe the effect of transient heating or cooling of the lower boundary. Such a process must occur initially to obtain a uniform thermal gradient across the layer. Reformulating the problem in variational form they show that $|\beta_c|$ varies significantly with δ, there being a 20 per cent decrease as δ varies from 1 to about 0·7, where it takes its smallest value. This behaviour is similar to that found by Nield [34] and Currie [35] for newtonian fluids.

6. Further studies in stationary convection

The effect of a change in the initial equilibrium alignment upon threshold values and the type of roll instability expected is of some interest. To this end Barratt and Sloan [36] consider the onset of stationary convection in a horizontal layer where the initial equilibrium configuration is a twisted orientation pattern in which the anisotropic axis lies parallel to the bounding plates but changes direction uniformly with distance between them. Seeking solutions of the form

$$v = (v_1, v_2, iv_3)\exp i(lx + my), \quad \hat{\mathbf{n}} = (in_1, in_2, n_3)\exp i(lx + my),$$
$$\hat{s} = is\exp i(lx + my), \tag{6.1}$$

where v_i, n_i, s are functions of z, they obtain results by direct numerical integration of the resulting linear differential equations using orthornormalization. Threshold gradients are found to be rather insensitive to the amount of twist in the sample with values being similar to those obtained for the planar problem. However the shape of the cellular instability pattern changes appreciably with twist so that at the onset of convection $m/l \sim \tan(\theta_t/2)$, where θ_t is the total twist in the layer. The parameter $\beta_c h^4$ appears to be almost a universal constant.

Barratt and Bramley [37] examine convection in obliquely aligned nematic layers where the initial uniform orientation of the director lies in the xz plane and makes an angle θ_0 with the positive x axis. They investigate the onset of two particular types of instability in 1 mm thick layers. One is a Y axis roll with

$$\hat{\mathbf{n}} = i\{n_1, 0, n_3\}\exp(imx), \mathbf{v} = \{v_1, 0, v_3\}\exp(imx), \hat{s} = s\exp(imx) \tag{6.2}$$

and the other is an X axis roll with

$$\hat{\mathbf{n}} = \{n_1, in_2, n_3\}\exp(imy), \mathbf{v} = \{v_1, iv_2, v_3\}\exp(imy), \hat{s} = s\exp(imy), \tag{6.3}$$

where n_i, v_i and s are functions of z. In the absence of a magnetic field, they predict a Y roll instability when heating is from below and an X roll instability when heating is from above. Threshold magnitudes increase as θ_0 moves away from 0 or $\pi/2$ when heating is from below or above, respectively, implying that the heat focusing effect decreases as the tilt from the planar or homeotropic alignment increases. Again $\beta_c h^4$ is found to be almost a universal constant. Investigating the effect of a stabilizing magnetic field in 5 mm layers, Barratt and Bramley [38] show that, when heating is from above, a transition from an X roll to a Y roll occurs at a value $\theta_T(H)$ of the tilt angle θ_0. The reason for this transition is the inhibiting effect on the onset of instability of a magnetic torque associated with the three dimensional nature of an X roll which is absent in the two dimensional Y roll. These predictions agree tolerably well with the very limited experimental evidence provided by Guyon, Pieranski and Boix [39]. However they observe a pairing of rolls with different wavelengths which results in a pattern of alternate wide and narrow cells and, as yet, no quantitative analysis of this phenomenon has been attempted.

A free surface effect of some importance arises in the so called Bénard–Marangoni problem. In this the upper boundary is a free surface where a thermal fluctuation induces a surface traction arising from the variation of the surface tension σ with temperature. This phenomenon has a destabilizing influence upon the system and is called the Marangoni effect. Employing the one dimensional analysis of §4, Guyon and Velarde [40] show that, in general, critical values of the Marangoni number M and the Rayleigh number R satisfy the condition

$$\frac{M}{M_c(R = 0)} + \frac{R}{R_c(M = 0)} = 1. \tag{6.4}$$

where

$$M \equiv -\beta h^2(\partial\sigma/\partial T)/(k_\parallel \tilde{\eta}),$$

$$R \equiv -\alpha' \varrho g \beta h^4/(k_\parallel \eta_b)$$

and $\tilde{\eta}$ is an average viscosity. Velarde and Zuniga [27] investigate the stability of planar and homeotropic layers with respect to disturbances of the form in equations (5.1) and (5.2), respectively, and obtain results for critical values that agree well with the relation (6.4). Urbach, Rondelez, Pieranski and Rothen [41] report observations of the Marangoni effect in nematic droplets subject to heating by a hot spot and present a theoretical calculation of this related problem which seems to be in good agreement with the observations.

A problem related to that of Rayleigh–Bénard convection is one in which a thermal gradient is applied radially across a sample of nematic liquid crystal contained in a cylindrical annulus rotating with constant angular velocity ω_0 about its vertical axis. Here the effect of the centrifugal buoyancy force is rather similar to that of the gravitational buoyancy force in the Bénard problem with $\omega_0^2 r$ replacing g. Carrigan and Guyon [42] describe such an experiment and when $\mathbf{n}^0$ is radial their observations are similar to those for the gravitational instability with the axis of the rolls being vertical. When $\mathbf{n}^0$ is vertical they demonstrate the influence of the Coriolis force upon the alignment of the roll axis, this axis being vertical when the Coriolis force is dominant. They also report a first order phase transition with hysteresis effects in this case. Using the narrow gap approximation and assuming the annulus to be very long, Barratt and Zuniga [43] give a detailed numerical linear stability analysis of this latter arrangement. Their results suggest that a Taylor–Couette instability (azimuthal rolls) rather than the Taylor–Proudman instability (axial rolls) occurs, which seems to be in conflict with the observations of Carrigan and Guyon [42]. Although reasons are suggested for this apparent conflict, it would appear that further analysis and experimentation is necessary to clarify the situation.

Finally Horn, Guyon and Pieranski [44], Kini [45] and Guyon, Pieranski and Boix [39] examine the effect of free convection in tilted planar and homeotropic layers.

7. Oscillatory convection in homeotropic nematics

The onset of stationary convection in homeotropic nematics heated from above is driven by a strongly destabilizing heat focusing effect. An obvious question which poses itself is whether convection is possible when heating is from below and the heat focusing effect is strongly stabilizing. Of course, in this event, there is the usual destabilizing buoyancy force due to the presence of a downward thermal gradient. In the absence of an applied magnetic field, equation (4.8) indicates that the relaxation time for director perturbations T_n is much greater than that for thermal fluctuations T_s. Motivated by this observation, Lekkerkerker [46] was the first to investigate the possibility of oscillatory convective modes in nematic layers.

Considering a flow field in which

$$v_z = v(\mathbf{q}) \cos \omega t \cos q_x x \sin q_z z, \tag{7.1}$$

where the frequency ω satisfies the inequality

$$K_3 q^2 \ll \omega \ll (\kappa_1 + \kappa_2) q^2, \tag{7.2}$$

it follows from the field equations that the corresponding non-zero components in the perturbations of alignment and temperature have the form

$$\hat{n}_x = n(\mathbf{q}, \omega) \sin \omega t \sin q_x x \sin q_z z, \quad \hat{s} = s(\mathbf{q}) \cos \omega t \cos q_x x \sin q_2 z. \tag{7.3}$$

It should be noted that the inequality (7.2) requires the relaxation time T_s to be small compared with the relaxation time T_n. When this is the case it follows from equations (7.1) and (7.3) that fluctuations in flow and temperature remain in phase while those in alignment lag 90° behind. This allows the possibility of a distortion to be initiated when the destabilizing buoyancy force exceeds the stabilizing viscous torque which is expected to occur at a threshold comparable with that for stationary convection in isotropic liquids. However, as time progresses the strongly stabilizing heat focusing force acts to restore the initial alignment and in the process the director overshoots its equilibrium value and an oscillatory motion results. It is the competition between a stabilizing effect with a long relaxation time and a destabilizing effect with a short relaxation time that allows the possibility of oscillatory or overstable convective modes and this is due to the mechanism of dephasing the two effects. This phenomenon of dephasing a stabilizing effect and thus producing the possibility of overstability is present in other systems, as is illustrated by the Soret effect in binary mixtures [47].

Seeking a solution in which perturbation variables are expressed as a superposition of Fourier modes, Lekkerkerker [46] was the first to predict that the principle of exchange of stabilities does not always hold for nematic liquid crystals in the presence of a vertical thermal gradient. He also indicates that a good approximation for the overstable threshold β_c^0 is given by

$$\varrho g \alpha' \beta_c^0 h^4 = \nu k q^6 h^4 / q_x^2, \tag{7.4}$$

where ν is some average viscosity and

$$k = (k_{\parallel} q_z^2 + k_{\perp} q_x^2)/q^2,$$

this being identical to the relation which yields the threshold for stationary convection in an isotropic liquid with viscosity ν and thermal diffusivity k. Setting $q_x = q_z = \pi/h$ in equation (7.4) suggests that observation of oscillatory convection necessitates the use of comparatively thick layers with $h = 5$ mm. It is, therefore, not surprising that Pieranski, Dubois-Violette and Guyon [20] did not detect overstability in their experiments employing significantly thinner layers.

Experimental support for Lekkerkerker's predictions was soon provided by Guyon, Pieranski and Salan [48]. Heating 5 mm thick layers of MBBA from below, they investigated the stabilizing effect of a vertical magnetic field upon both the threshold gradient and the nature of the instability. By increasing the magnetic field the relaxation time T_n for fluctuations in orientation is effectively decreased by several orders of magnitude until T_n and T_s are of the same order. The heat focusing mechanism is then no longer dephased and the onset of stationary convection becomes a possibility. Employing the simple one dimensional analysis described in §4, Guyon, Pieranski and Salan [48] predict that the frequency of oscillations is a quadratic function of B^2 which vanishes when the flux density is $B_T \approx 0{\cdot}058$ T, whereupon a transition from oscillatory to stationary convection results. The model also predicts that $|\beta_c|$ increases linearly with B^2 until the transition occurs and then decreases. The predictions of this simple model agree remarkably well with their experimental observations and also agree qualitatively with Lekkerkerker's results. However the experiments also exhibit the interesting phenomena of hysteresis effects and the occurrence of an inverse bifurcation. Unfortunately linear theory is unable to provide any information on these matters and a discussion of such topics is postponed to the next section.

Rather more detailed linear analyses of overstability given by Lekkerkerker [49], Velarde and Zuniga [27], Barratt and Sloan [50] and Barratt and Manley [51] yield substantially the same qualitative results as those provided by the one dimensional analysis. However they also predict that for very large magnetic fields the heat focusing effect is virtually eliminated and the critical threshold attains an asymptotic value. Barratt and Bramley [38] allow for overstable modes in a detailed numerical investigation of convective instabilities in obliquely aligned 5 mm thick layers. Allowing for the effect of a stabilizing magnetic field, they show that only stationary convection is possible when heating is from below. However, when heating is from above the situation is rather complex, especially for values of θ_0 between 50° and 80°. Here there are a variety of possibilities with transitions between X and Y roll instabilities and stationary and oscillatory instabilities.

8. A non-linear approach

Although linear analyses can predict critical values at which an instability is possible they provide no information as to how the instability develops as the threshold is exceeded. It is also obvious that they cannot decide whether finite amplitude perturbation solutions are possible for temperature differences smaller than those predicted by the linear theory itself. That is they cannot distinguish between the possibility of subcritical and supercritical phenomena. Such information can only be obained by retaining the non-linear terms in the analysis.

The relatively few non-linear analyses so far attempted all seek two dimensional solutions of equations (2.1)–(2.4) in which

$$\mathbf{v} = (u, 0, v), \quad T = T_l + \beta z + s \quad \text{and} \quad \mathbf{n} = (\cos\theta, 0, \sin\theta) \quad \text{or} \quad (\sin\theta, 0, \cos\theta), \tag{8.1}$$

where θ is the angle between the director and the initial alignment. The problem is to solve a set of partial differential equations

$$L\mathbf{U} = N\mathbf{U}, \tag{8.2}$$

where $\mathbf{U} \equiv (u, v, \theta, s)^T$, L is a linear and N a non-linear operator. Utilizing the series expansions

$$\mathbf{U} = \sum_{i=1}^{\infty} \mathbf{U}^{(i)}\varepsilon^i, \quad \beta = \beta_c + \sum_{i=1}^{\infty} \beta_i \varepsilon^i, \tag{8.3}$$

where $\mathbf{U}^{(i)} \equiv (u_i, v_i, \theta_i, s_i)^T$, β_i is constant and ε is a small parameter, in equation (8.2) yields a set of partial differential equations

$$L(\beta_c)\mathbf{U}^{(1)} = \mathbf{0}, \quad L(\beta_c)\mathbf{U}^{(m)} = \mathbf{Y}^{(m)}, \quad m \geqslant 2. \tag{8.4}$$

Here $\mathbf{U}^{(1)}$ is the solution of the corresponding linear problem and $\mathbf{Y}^{(m)}$ represents the non-linear terms. The adjoint operator L^* is defined by the relation

$$\langle L\mathbf{U}, \mathbf{U}^* \rangle = \langle \mathbf{U}, L^*\mathbf{U}^* \rangle, \tag{8.5}$$

where $\langle . , . \rangle$ denotes a suitable inner product. Hence, if $L^*(\beta_c)\mathbf{U}^* = \mathbf{0}$, the Fredholm alternative yields

$$\langle \mathbf{Y}^{(m)}, \mathbf{U}^* \rangle = 0 \tag{8.6}$$

as a necessary condition for the existence of a bounded solution of equation (8.4) and it is this condition which determines the nature of the bifurcation.

The first attempt at a non-linear analysis was presented by Dubois-Violette and Rothen [52] for the onset of stationary convection in planar samples. Since the corresponding linear problem does not have an amenable analytic solution, they consider the so called free-free problem with stress free boundaries. Although this is unrealizable in practice it has the advantage of having a rather simple analytic solution. After determining β_c and the solution of the adjoint problem, they employ equation (8.6) with $m = 2$ and 3 to show that $\beta_1 = 0$ and $\beta_2 > 0$, respectively, for a wide range of viscosity values including the case close to a smectic-nematic transition. Thus they predict that bifurcation is normal and the instability is supercritical for the planar problem. A similar analysis, by Dubois-Violette and Gabay [53], for convection in homeotropic layers heated from below allows for the possibility of overstability. In the absence of a stabilizing magnetic field, they predict the onset of oscillatory convection and find that $\beta_1 = 0$ and $\beta_2 < 0$. This means that the instability is subcritical with finite amplitude solutions being possible at thresholds smaller than those predicted by the linear theory. These results are also in accord with the predictions of Lekkerkerker [46, 49] and the observations of Guyon, Pieranski and Salan [48] concerning the occurrence of oscillatory convection and inverse bifurcation in homeotropic samples heated from below. Investigating the effect of a stabilizing magnetic field Dubois-Violette and Gabay [53] find that β_2 remains negative until B exceeds a value $B_0 < B_T$ when it becomes positive and a normal bifurcation is expected.

Hodson, Barratt and Sloan [54, 55] present more detailed investigations of both problems and obtain accurate numerical solutions which satisfy the more realistic boundary conditions (3.10). Their results agree qualitatively with those just described and it is shown, in all analyses, that the disturbance amplitude is proportional to $|\beta - \beta_c|^{1/2}$, for sufficiently small ε. It must be emphasized that the results obtained using this perturbative expansion method of solution are only valid provided the point (β, a) is sufficiently close to (β_c, a_c). In addition Hodson, Barratt and Sloan [55] indicate that the method appears to be invalid in the region of $B = B_0$, where the coefficient β_2 is not defined.

Otnes and Riste [56, 57, 58] employ a neutron scattering method to investigate experimentally the multicritical behaviour of a homeotropic nematic heated from below in the presence of a stabilizing magnetic field. In obtaining a fairly detailed picture of the phase diagram in $(\Delta T, B)$ space, they predict a rather more complex situation than that presented here and suggest there is some conflict between these observations and the conclusions of the previous authors [53, 55]. Since some support for the conclusions of Otnes and Riste is forthcoming from the work of Knobloch, Weiss and Da Costa [59] on magnetoconvection in a Boussinesq fluid, it is clear that more detailed analysis and experimentation is required to resolve the situation.

9. Convective instabilities in cholesterics

Investigations of convective phenomena in cholesteric liquid crystals are considerably less numerous than those pertaining to nematics. This is undoubtedly due to the complexities introduced into the equations by their inherent helical structure and the additional thermomechanical coupling terms allowed by the weaker symmetries associated with them. Lehmann [60] first reported a coupling between a thermal gradient and orientation, and Leslie [9] provided a subsequent explanation. Leslie [9, 61] analyses two possible arrangements for determining the importance of the material parameters γ_3, α_7 and α_8 which are associated with the direct coupling of orientation and thermal gradient, while Eber and Janossy [62, 63] describe a method for measuring γ_3. Prost [14] suggests that the extra coupling parameters are relatively unimportant. Apart from these studies little attempt has been made to obtain estimates for these material parameters.

To avoid complications in the Ericksen–Leslie formulation due to the spatial dependence of a uniformly twisted equilibrium configuration, the earlier investigations [64, 65, 66] confine attention to short pitch cholesterics ($P \ll h$) and employ a linear theory proposed by Martin, Parodi and Pershan [67] and Lubensky [68] to examine the onset of convection in horizontal layers. They also consider that the anisotropic axis lies in the horizontal plane initially, and that at least one bounding surface is a free surface. Assuming the pitch to be independent of temperature and ignoring the extra coupling terms associated with cholesterics, Dubois-Violette [64] obtains an expression which determines the critical thermal gradient for the onset of stationary convection. For a given pitch, it indicates that β_c is now inversely proportional to h^2 rather than h^4. This means that for critical temperature differences of only a few degrees relatively thick layers ($h = 1$ cm) and short pitches (10–100 μ) are required. She also demonstrates that the heat focusing mechanism drives the stationary instability with the sign of k_a again being crucial.

Parsons [65] seeks normal mode solutions and employs free–free boundary conditions to derive a dispersion relation between mode frequencies and their wavevectors.

With the assumption that the pitch is independent of temperature, he examines the effect of a stabilizing magnetic field applied parallel to the helical axis and effectively recovers Dubois-Violette's result in the limiting case as B tends to zero. Allowing the pitch to be weakly dependent upon temperature, Parsons shows that the convective instability is oscillatory and describes its driving mechanism in terms of an exchange of kinetic energy associated with the destabilizing buoyancy forces and the potential energy associated with the elastic restoring forces. In presenting a more detailed analysis, Pleiner and Brand [66] allow for the effects of the extra terms peculiar to cholesterics. Their inclusion allows additional mechanisms to affect the onset of instability, the effects of which are not easy to isolate. However calculations suggest that when they are relevant the instability is oscillatory. Finally, Dubois-Violette and de Gennes [69], Pleiner and Brand [66] and Ranganath [70] discuss some aspects of convection in relation to so called permeation effects.

Later studies [31, 71, 72] employ the Ericksen–Leslie theory to investigate convection in comparatively long pitch cholesterics. These examine stability with respect to a wider class of infinitesimal disturbances which also satisfy the more realistic boundary conditions (3.10). Fraser [31] eliminates complications due to the inherent helical structure of a cholesteric by considering convection in uniformly aligned planar or homeotropic layers which, as Fischer [73] demonstrates, represent possible equilibrium configurations between parallel plates provided $P \geqslant 4h$. Neglecting all material parameters peculiar to cholesterics except τ, Fraser considers stationary perturbation solutions of the form

$$\hat{\mathbf{n}} = (n_1, n_2, 0)\exp(imx), \quad \mathbf{v} = (v_1, 0, v_3)\exp(imx), \quad \hat{s} = s\exp(imx) \qquad (9.1)$$

and

$$\hat{\mathbf{n}} = (0, n_1, n_2)\exp(imx), \quad \mathbf{v} = (v_1, 0, v_3)\exp(imx), \quad \hat{s} = s\exp(imx) \qquad (9.2)$$

for homeotropic and planar layers, respectively, where n_i, v_i and s are functions of z alone. Adopting Jeffrey's Fourier series method [25] to solve the problem, he shows that $\beta_c h^4$ is a universal function and demonstrates that the chirality introduces a weakly destabilizing mechanism which is quantitatively similar to the effect of an induced twist in twisted nematic layers [36].

In a later paper, Fraser [71] considers stationary convection in a uniformly twisted long pitch cholesteric. Dafermos [74] argues that a likely equilibrium solution is

$$\mathbf{n}^0 = (\cos\theta(z), \sin\theta(z), 0), \quad \theta = \theta_t z/h, \quad T = T_1 + \beta z, \qquad (9.3)$$

where θ_t is the total twist imposed on the sample, provided τ satisfies the inequality $h\tau - \pi/2 < \theta_t < h\tau + \pi/2$. Seeking perturbation solutions of the form in equation (6.1), Fraser obtains accurate numerical results by direct integration of the governing ordinary differential equations and finds that in all cases convection occurs when heating is from below. For fixed values of the inherent twist $h\tau$ and the imposed twist θ_t, the equations indicate that $\beta_c h^4$ is a universal function. Results for values of $h\tau$ between 0 and 2π demonstrate a significantly wider range of values for β_c than those reported for a twisted nematic [36] or uniformly aligned cholesteric [31]. In fact for a given pitch the variation of β_c with θ_t can be quite remarkable. As for a twisted nematic, $m/l \approx \tan(\theta_t/2)$ at threshold. However calculations indicate that this relation is not valid for values of $h\tau$ in excess of 2π.

Barratt and Hodson [72] allow for the possibility of overstability when examining the effect of material parameters peculiar to cholesterics upon the onset of convection in homeotropic layers. The flow field is no longer restricted to the xz-plane as in equation (9.1) and they consider solutions of the form

$$\mathbf{v} = (v_1 \sin ax,\ v_2 \cos ax,\ v_3 \cos ax) \exp(-i\omega t),\ \hat{\mathbf{n}} = (n_1,\ n_2,\ 0) \sin ax \exp(-i\omega t),$$

$$\hat{s} = s \cos ax \exp(-i\omega t), \tag{9.4}$$

where v_i, n_i and s are functions of z. Computation of numerical solutions using Chebyshev collocation suggests that oscillatory convection occurs in 5 mm thick layers when heating is from below and that stationary convection normally occurs in 1 mm thick layers heated from above. In both cases it appears that threshold values are relatively insensitive to changes in α, τ, κ_3 and κ_4. However, the effect of direct coupling between the thermal gradient and orientation through the parameters α_7, α_8 and γ_3 is strongly destabilizing when heating is from below with a 50 per cent reduction in $|\beta_c|$ as the parameters vary from 10^{-4} to 10^{-2} in c.g.s. units. The effect is somewhat more complicated when heating is from above, but a strongly stabilizing influence is demonstrated by a 70 per cent increase in $|\beta_c|$ as $|\alpha_7|$ and $|\alpha_8|$ increase from 10^{-6} to 10^{-5} when α_7 and α_8 are of opposite sign. Finally we note that investigations by Eber and Janossy [62, 63] suggest that the estimates for α_7, α_8 and γ_3 employed in calculations by Barratt and Hodson [72] are not necessarily unreasonable.

10. Some concluding remarks

The linear theory describing the onset of stationary and oscillatory convection in both planar and homeotropic layers of a nematic liquid crystal has received considerable attention and its predictions agree rather well with the somewhat limited experimental observations so far available. However, it is unfortunate that all the experimental results relate to the rather unstable nematic material MBBA and it would be useful to confirm the predictions using one of the more stable nematic materials now available. With the abundance of theoretical results concerning the onset of convection in nematic layers, the paucity of experimental investigations is somewhat disappointing. For example there has been little or no attempt to confirm predictions concerning the effect of magnetic fields on thresholds in planar and homeotropic samples or results pertaining to twisted and tilted nematic layers. Experimental verification of theoretical predictions would serve as a check on the validity of the investigations whereas any discrepancy would suggest that we examine instability with respect to more general disturbances than those employed or question various assumptions in the analysis. For example, the possible importance of non-Boussinesq effects has so far been neglected. A study incorporating these would be useful when material parameters vary significantly with temperature, as is the case near a smectic-nematic transition, or when relatively large temperature differences are required for the onset of convection.

The rather complex non-linear equations have, as yet, received comparatively little attention and those analyses attempted confine themselves to seeking results that are only valid close to the threshold value predicted by linear theory. Although such calculations support the possibility of inverse bifurcation in homeotropic layers heated from below they do not describe how the instability develops beyond the vicinity of the linear threshold nor can they predict possible further transitions above this threshold. In the presence of stabilizing magnetic fields, homeotropic nematics exhibit steady convection preceded by an oscillatory regime due to the existence of competing relaxation times. They represent, therefore, a physical system which exhibits multicritical phenomena and a detailed study of this system might well lead to a fuller understanding of multicritical behaviour in similar and even more complicated physical systems. Such an investigation might also provide some understanding of chaotic behaviour in non-equilibrium systems and the development of turbulence which is currently of great physical interest. For these reasons more sophisticated analyses of the non-linear equations are desirable.

Although there are a variety of predictions concerning the onset of convection in cholesteric materials, there appears to be very little in the literature concerned with corresponding experimental investigations. This is particularly disappointing since predictions suggest that such investigations could readily determine the relative importance of the thermomechanical coupling terms peculiar to long pitch cholesterics. Before more detailed theoretical studies are undertaken it seems desirable to have experimental evidence which lends support to the continuum theory utilised in the analyses presented so far.

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